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Stabilization of the motions of mechanical systems in prescribed phase-space manifolds☆

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Abstract

A method for constructing a mathematical model of the dynamics of a mechanical system is proposed. An algorithm is constructed for determining the expressions for the control forces and the components of the constraint reactions. A modification is made to the dynamic equations which enables one to solve the problem of stabilizing the constraints and which ensures the required accuracy in the numerical solution of the corresponding system of differential-algebraic equations describing the constraints imposed on the system, its kinematics and dynamics. By virtue of well-known dynamic analogies, the proposed method can be used to investigate the dynamics of different physical systems. The problem of modelling the dynamics of an adaptive optical system with two degrees of freedom is considered.

1. Introduction

The dynamic equations of a mechanical system can be constructed if we know its kinetic energy $T^0 = T^0(q^i, \dot{q}^j)$, $\dot{q} \triangleq dq/dt$ $(i, j = 1, 2, \ldots, n)$, potential energy $P^0 = P^0(q^i, t)$, dissipative function $D^0 = D^0(q^i, \dot{q}^j, t)$, the elementary work of the generalized non-potential forces $Q_s = Q_s(q^i, \dot{q}^j, t)$ and the control forces $R_s = R_s(q^i, \dot{q}^j, t)$ $(s = 1, 2, \ldots, n)$.

$$\delta A = Q_s \delta q^s + R_s \delta q^s \tag{1.1}$$

In equality (1.1) and henceforth, summation over repeated indices is assumed. Using well-known dynamical analogies,[1,2] the corresponding reasoning can also be used for physical systems. The control forces R_s acting on the system are called upon to ensure that the constraint equations

$$\begin{gathered} f^\mu(q^i, t) = 0, \quad \partial_s f^\mu \dot{q}^s + \partial_t f^\mu = 0, \quad f^\rho(q^i, \dot{q}^j, t) = 0 \\ \partial_s f^\mu \triangleq \frac{\partial f^\mu}{\partial q^s}, \quad \partial_t f^\mu \triangleq \frac{\partial f^\mu}{\partial q^s}; \quad \mu = 1, \ldots, m, \quad \rho = m+1, \ldots, r \end{gathered} \tag{1.2}$$

which are imposed on the generalized coordinates q^i and velocities $\dot{q}^j$ of the system are satisfied. The left-hand sides of Eq. (1.2) are differentiable with respect to all the variables.

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The well-known classical methods for constructing the dynamic equations are based on the assumption that the constraint equations are satisfied when $t = t_0$

$$f^{\mu}(q_0^i, t_0) = 0, \quad \partial_s f^{\mu}(q_0^i, t_0)\dot{q}_0^s + \partial_t f^{\mu}(q_0^i, t_0) = 0, \quad f^{\rho}(q_0^i, \dot{q}_0^j, t_0) = 0$$
$$q^i(t_0) = q_0^i, \quad \dot{q}^j(t_0) = \dot{q}_0^j \tag{1.3}$$

and for all $t > t_0$. If, however, the initial conditions (the last two equalities of (1.3)) turn out to be such that

$$f^{\mu}(q_0^i, t_0) = c^{\mu}, \quad \partial_s f^{\mu}(q_0^i, t_0)\dot{q}_0^s + \partial_t f^{\mu}(q_0^i, t_0) = c^{m+\mu}, \quad f^{\rho}(q_0^i, \dot{q}_0^j, t_0) = c^{m+\rho}$$

$$c^{\chi} = \text{const}, \quad \chi = 1, \dots, m + r$$

then the numerical solution of the differential-algebraic equations composed of the kinematic equations, the dynamic equations and the constraint equations turns out to be unstable, and the deviation from the constraint equations increases with time. In order to stabilize the constraints (1.2), it is necessary to take account of the deviation from Eq. (1.2) and to introduce a corresponding correction into the right-hand sides of the dynamic equations of the system.[3,4] In recent years, the problem of stabilizing the constraints and constructing stable difference schemes for solving the differential-algebraic equations has become an urgent problem in modelling the dynamics of mechanical systems.[5,6]

A method of constructing a mathematical model of the dynamics is proposed below and an algorithm for determining the reactions of the constraints R_s, which ensure that the constraint Eq. (1.2) are satisfied, is constructed. A modification is made to the dynamic equations which enables the problem of stabilizing the constraints to be solved and ensures the required accuracy in the numerical solution of the system of differential-algebraic equations describing the constraints imposed on the system, its kinematics and dynamics.

2. Construction of the dynamic equations

Additional parameters, the excess variables y^{μ}, $\dot{y}^{\mu}$, $\dot{y}^{\rho}$, which are henceforth labelled using letters from the Greek alphabet, are introduced into the treatment in order to estimate the deviations from the constraint equations (1.2) by means of the equalities

$$y^{\mu} = f^{\mu}(q^i, t), \quad \dot{y}^{\mu} = \partial_s f^{\mu}\dot{q}^s + \partial_t f^{\mu}, \quad \dot{y}^{\rho} = f^{\rho}(q^i, \dot{q}^j, t) \tag{2.1}$$

Taking account of these new variables, the kinematic state of the system corresponding to the mathematical model will be determined by the generalized coordinates q^i, y^{μ} and the generalized velocities $\dot{q}^i$, $\dot{y}^{\kappa}$ ($\kappa = 1, \dots, r$). The kinetic energy, potential energy and dissipative function will also contain the excess coordinates y^{μ} and velocities $\dot{y}^{\kappa}$: $T = T(q^i, y^{\mu}, \dot{q}^j, \dot{y}^{\kappa})$, $P = P(q^i, y^{\mu}, t)$ $D = D(q^i, y^{\mu}, \dot{q}^j, \dot{y}^{\kappa}, t)$. It is assumed that the functions T, P and D are at least doubly differentiable with respect to all of the variables and that the conditions

$$T = T^0(q^i, \dot{q}^j), \quad \partial_{\mu}T = 0, \quad \dot{\partial}_{\kappa}T = 0, \quad \partial^2_{\mu\nu}T = 0, \quad \partial_{\mu}\dot{\partial}_{\kappa}T = 0, \quad \dot{\partial}^2_{\kappa\eta}T = a_{\kappa\eta}(q^i)$$

$$\partial_{\mu}T \triangleq \frac{\partial T}{\partial y^{\mu}}, \quad \dot{\partial}_{\kappa}T \triangleq \frac{\partial T}{\partial \dot{y}^{\kappa}}, \quad \partial^2_{\mu\nu}T \triangleq \frac{\partial^2 T}{\partial y^{\mu}\partial y^{\nu}}, \quad \partial_{\mu}\dot{\partial}_{\kappa}T \triangleq \frac{\partial^2 T}{\partial y^{\mu}\partial \dot{y}^{\kappa}}, \quad \dot{\partial}^2_{\kappa\eta}T \triangleq \frac{\partial^2 T}{\partial \dot{y}^{\kappa}\partial \dot{y}^{\eta}}$$

$$P = P^0(q^i, t), \quad \partial_{\mu}P = 0, \quad \partial^2_{\mu\nu}P = k_{\mu\nu}(q^i, t)$$

$$D = D^0(q^i, q^j, t), \quad \dot{\partial}_{\kappa}D = 0, \quad \dot{\partial}^2_{\kappa\eta}D = c_{\kappa\eta}(q^i, \dot{q}^j, t)$$

$$\nu = 1, \dots, m, \quad \eta = 1, \dots, r$$

are satisfied when

$$y^{\mu} = 0, \quad \dot{y}^{\kappa} = 0 \tag{2.2}$$

If the values of the variables y^μ and $\dot{y}^\kappa$ are sufficiently small: $||z|| \le \varepsilon$, $z = (y^\mu, \dot{y}^\kappa)$, then, on putting

$$T^0 = \frac{1}{2}m_{ij}(q^s)\dot{q}^i\dot{q}^j \tag{2.3}$$

the functions T, P and D can be represented by an expansion in series in the powers of y^μ and $\dot{y}^\kappa$:

$$T = \frac{1}{2}m_{ij}(q^s)\dot{q}^i\dot{q}^j + \frac{1}{2}a_{\kappa\eta}(q^s)\dot{y}^\kappa\dot{y}^\eta + T^{(3)} \tag{2.4}$$

$$P = P^0(q^s, t) + \frac{1}{2}k_{\mu\nu}(q^s, t)y^\mu y^\nu + P^{(3)} \tag{2.5}$$

$$D = D^0(q^s, \dot{q}^k, t) + \frac{1}{2}c_{\kappa\eta}(q^s, \dot{q}^k, t)\dot{y}^\kappa\dot{y}^\eta + D^{(3)} \tag{2.6}$$

Here, $T^{(3)}$, $P^{(3)}$, $D^{(3)}$ are the corresponding terms which contain the factors of y^μ and $\dot{y}^\kappa$ to powers of no less than three. It is assumed that the coefficients m_{ij}, $a_{\kappa\eta}$, $k_{\mu\nu}$, $c_{\kappa\eta}$ and all of their partial derivatives are bounded in the domain Ω of the change of variables q^s, $\dot{q}^j$ and for all $t \ge t_0$. The forces R_s correspond to the coordinates q^s and are considered as control forces which ensure that the equalities (2.1) are satisfied.

The d'Alembert - Lagrange principle

$$\begin{gathered}
\mathscr{E}_i\delta q^i + \mathscr{E}_{n+\kappa}\delta y^\kappa = 0 \\
\mathscr{E}_i \triangleq E_i(T) + \partial_i P + \dot{\partial}_i D - Q_i - R_i \\
\mathscr{E}_{n+\kappa} \triangleq E_\kappa(T) + \partial_\kappa P + \dot{\partial}_\kappa D - Q_\kappa - R_\kappa \\
E_i(T) \triangleq (\dot{\partial}_i T)^{\cdot} - \partial_i T, \quad E_\kappa(T) \triangleq (\dot{\partial}_\kappa T)^{\cdot} - \partial_\kappa T, \quad (\dot{\partial}_i T)^{\cdot} \triangleq \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i}, \quad (\dot{\partial}_\kappa T)^{\cdot} \triangleq \frac{d}{dt}\frac{\partial T}{\partial \dot{y}^\kappa}
\end{gathered} \tag{2.7}$$

can be used for the construction of the system of differential equations which corresponds to the expressions (2.4)–(2.6) which have been adopted for the functions T, P and D.

If the variations in the excess variables δy^κ are determined from the last two equations of (2.2) according to the rule

$$\delta y^\kappa = \dot{\partial}_i \dot{y}^\kappa \delta q^i; \quad \dot{\partial}_i \dot{y}^\mu = \partial_i f^\mu, \quad \dot{\partial}_i \dot{y}^\rho = \dot{\partial}_i f^\rho \tag{2.8}$$

then the possible displacements δq^i of the system must be determined by the solution of the system of r linear algebraic equations (2.8) in the n unknowns.

The general solution of system (2.8) consists of two terms:

$$\delta q^i = v^i \delta s + f_\kappa^{(+)i}\delta y^\kappa \tag{2.9}$$

The first term corresponds to a direction which is tangential to the manifold Φ in phase space, defined by Eq. (2.1) in the case of fixed values of the variables y^μ and $\dot{y}^\rho$. The quantity v^i is the corresponding component of the vector product $v = [FC]$ of a row of the matrix $F = (f_i^\kappa)$, $f_i^\kappa \triangleq \dot{\partial}_i f^\rho$ and a row of an arbitrary matrix $C = (c_j^\beta)$ $(\beta = r+2, \ldots, n)$, and δs is an arbitrarily small quantity. The component v^i is calculated as the determinant of the matrix formed by the unit vector $(\delta_1^i, \ldots, \delta_n^i)$, $(\delta_j^i = 0, i \ne j, \delta_i^i = 1)$ and the rows of the matrices F and C. The second term corresponds to a direction which is normal to the manifold Φ. The coefficients $f_\kappa^{(+)i}$ constitute the matrix $F^+ = F^T(FF^T)^{-1}$, where F^T is the transpose of the matrix F.

The magnitude of the elementary work of the generalized control forces, when account is taken of expression (2.9) for the virtual displacements of the system, is given by the sum

$$R_i\delta q^i = R_i v^i \delta s + R_i f_\kappa^{(+)i}\delta y^\kappa \tag{2.10}$$

If the control forces R_i are chosen from the set of functions satisfying the equality

$$R_i v^i = 0 \tag{2.11}$$

then expression (2.10) takes the form

$$R_i\delta q^i = \lambda_\kappa \delta y^\kappa \tag{2.12}$$

Actually, in this case, $R_i v^i$ is the scalar (composite) product $\{RFC\} = R^T[FC]$ of the vector R and the vectors constituting the rows of the matrices F and C. The quantity $\{RFC\}$ is calculated as the determinant which is obtained by replacing the first row of the determinant v^i by the components $R_1, \ldots, R_n$ of the vector R. If equality (2.11) is possible for any matrix C, then the first $r+1$ rows of the determinant $\{RFC\}$ must be linearly dependent. This means that the vector $R = F^T\lambda$ is a linear combination of the row of the matrix F with arbitrary coefficients $\lambda_1, \ldots, \lambda_r$: $R_i = f_i^\kappa \lambda_\kappa$. This representation of the vector R of the control forces when $y^\mu = 0$, $\dot{y}^\kappa = 0$ corresponds to the reaction of ideal constraints in classical mechanics.

Taking equalities (2.9) and (2.12) into account, the expression for the d'Alembert - Lagrange principle (2.7) takes the form

$$\mathscr{E}_i(v^i\delta s + f_\kappa^{(+)i}\delta y^\kappa) - \lambda_\kappa\delta y^\kappa + \mathscr{E}_{n+\kappa}\delta y^\kappa = 0 \tag{2.13}$$

As a consequence of the independence of the variations δs and δy^κ, condition (2.13) in only satisfied when the following equalities hold

$$\mathscr{E}_i v^i = 0, \quad \mathscr{E}_i f_\kappa^{(+)i} - \lambda_\kappa + \mathscr{E}_{n+\kappa} = 0 \tag{2.14}$$

Since the first equality of (2.14) is analogous to condition (2.11), such sets λ_κ^* exist for which the following equality is satisfied

$$\mathscr{E}_i = f_i^\kappa \lambda_\kappa^* \tag{2.15}$$

Using relations (2.15) and $f_i^\kappa f_\eta^{(+)i} = \delta_\eta^\kappa$, it can be concluded from the second equality of (2.14) that $\lambda_\kappa^* = \lambda_\kappa$ and

$$\mathscr{E}_{n+\kappa} = 0 \tag{2.16}$$

Taking account of the notation adopted, equalities (2.15) and (2.16) can be represented in the form of a system of Lagrange equations

$$(\dot{\partial}_i T)^\cdot - \partial_i T = -\partial_i P - \dot{\partial}_i D + Q_i + f_i^\kappa \lambda_\kappa, \quad (\dot{\partial}_\kappa T)^\cdot - \partial_\kappa T = -\partial_\kappa P - \dot{\partial}_\kappa D \tag{2.17}$$

The system of differential-algebraic Eqs. (2.1), (2.17) enables us to determine the unknowns $q^i, \dot{q}^j, y^\mu, \dot{y}^\kappa, \lambda_\kappa$. The solution of this system reduces to the expression of the coefficients λ_κ and the excess variables y^μ and $\dot{y}^\kappa$ in terms of the generalized coordinates and velocities $q^i, \dot{q}^j$ and integration of the first system of Eq. (2.17) in the case of the specified initial conditions (1.3).

The expressions for λ_κ are determined if Eq. (2.17) are represented in a form which is resolved with reference to $\ddot{q}^i$ and $\ddot{y}^\kappa$. Taking account of expressions (2.4)–(2.6), we can write system (2.17) in the form

$$m_{ik}\ddot{q}^k = m_i + f_i^\kappa \lambda_\kappa - m_i^{(2)}, \quad a_{\kappa\eta}\ddot{y}^\eta = -b_{\kappa\eta}\dot{y}^\eta - k_{\kappa\eta}y^\nu - \mathscr{E}_{n+\kappa}^{(2)} \tag{2.18}$$

where

$$m_i = Q_i - \gamma_{i,jk}\dot{q}^j\dot{q}^k - P_i^0 - D_i^0, \quad \gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q^k} + \frac{\partial m_{ki}}{\partial q^j} - \frac{\partial m_{jk}}{\partial q^i}\right)$$

$$m_i^{(2)} = \frac{1}{2}((\partial_i a_{\kappa\eta} + \dot{\partial}_i c_{\kappa\eta})\dot{y}^\kappa\dot{y}^\eta + \partial_i k_{\mu\nu}y^\mu y^\nu) + \mathscr{E}_i^{(3)}$$

$$\mathscr{E}_i^{(3)} = E_i(T^{(3)}) + \partial_i P^{(3)} + \dot{\partial}_i D^{(3)}$$

$$\mathscr{E}_{n+\kappa}^{(2)} = E_{n+\kappa}(T^{(3)}) + \partial_\kappa P^{(3)} + \dot{\partial}_\kappa D^{(3)}$$

$$\partial_\rho P^{(3)} = 0; \quad b_{\kappa\eta} = \partial_i a_{\kappa\eta}\dot{q}^i + c_{\kappa\eta}$$

Denoting the matrices which are the inverses of (m_{is}) and $a_{\kappa\eta}$ by (m^{ki}) and $(a^{\xi\kappa})$ $(\xi = 1, \ldots, r)$ respectively, the system of Eq. (2.18) can be represented in a form which is resolved with reference to the higher derivatives:

$$\ddot{q}^k = m^k + f^{k\kappa}\lambda_\kappa + m^{k(2)} \tag{2.19}$$

$$\ddot{y}^\kappa = b_\eta^\kappa\dot{y}^\eta + k_\mu^\kappa y^\mu + Y^{\kappa(2)} \tag{2.20}$$

Here,

$$\begin{gathered} m^k = m^{ki}m_i, \quad f^{k\kappa} = m^{ki}\partial_i f^\kappa, \quad m^{k(2)} = -m^{ki}m_i^{(2)}, \quad b_\eta^\kappa = -a^{\kappa\xi}b_{\xi n}, \quad k_\mu^\kappa = -a^{\kappa\eta}k_{\eta\mu} \\ Y^{\kappa(2)} = -a^{\kappa\eta}\mathscr{E}_{n+\eta}^{(3)} \end{gathered} \tag{2.21}$$

3. Determination of the control actions

Expressions for the generalized control forces were constructed in the form $R_i = \partial_i f^\kappa\lambda_\kappa$ in Section 2. Equalities (2.1) have to be differentiated in order to determine the coefficients λ_κ, and we obtain

$$\begin{gathered} \ddot{y}^\kappa = \partial_k f^\kappa\ddot{q}^k + h_k^\kappa\dot{q}^k + h_t^\kappa \\ h_k^\mu = \partial_k\dot{f}^\mu, \quad h_t^\mu = \partial_t\dot{f}^\mu \\ \partial_k\dot{f}^\mu = \partial_{ki}^2 f^\mu\dot{q}^i + \partial_{kt}^2 f^\mu, \quad \partial_t\dot{f}^\mu = \partial_{it}^2 f^\mu\dot{q}^i + \partial_{tt}^2 f^\mu \\ \partial_{ki}^2 f^\mu \triangleq \frac{\partial^2 f^\mu}{\partial q^k\partial q^i}, \quad \partial_{kt}^2 f^\mu \triangleq \frac{\partial^2 f^\mu}{\partial q^k\partial t}, \quad \partial_{tt}^2 f^\mu \triangleq \frac{\partial^2 f^\mu}{\partial t^2} \\ h_k^\rho = \partial_k f^\rho, \quad h_t^\rho = \partial_t f^\rho \end{gathered} \tag{3.1}$$

Substituting the values of the generalized accelerations $\ddot{q}^k$ and $\ddot{y}^\kappa$ from (2.19) and (2.20) into equality (3.1) we obtain a system of linear algebraic equations for determining the Lagrange multipliers λ_κ

$$\begin{gathered} l^{\kappa\eta}\lambda_\eta = l^\kappa \\ l^{\kappa\eta} = \partial_k f^\kappa f^{k\eta}, \quad l^\kappa = l^{\kappa(0)} + l^{\kappa(1)} + l^{\kappa(2)} \\ l^{\kappa(0)} = -\partial_k f^\kappa m^k - h_k^\kappa q^k - h_t^\kappa, \quad l^{\kappa(1)} = b_\eta^\kappa\dot{y}^\eta + k_\mu^\kappa y^\mu, \quad l^{\kappa(2)} = Y^{\kappa(2)} - \partial_k f^\kappa m^{k(2)} \end{gathered} \tag{3.2}$$

The solution of system (3.2) can be represented in the form of a sum of three terms, distributed in powers of the variables y^μ and $\dot{y}^\kappa$:

$$\lambda_\kappa = \lambda_\kappa^{(0)} + \lambda_\kappa^{(1)} + \lambda_\kappa^{(2)}$$

$$\lambda_\kappa^{(0)} = l_{\kappa\eta} l^{\eta(0)}, \quad \lambda_\kappa^{(1)} = l_{\kappa\eta} l^{\eta(1)}, \quad \lambda_\kappa^{(2)} = l_{\kappa\eta} l^{\eta(2)}; \quad l_{\kappa\eta} l^{\eta\xi} = \delta_\kappa^\xi$$

With this choice of the coefficients λ_κ, which are distributed in powers of the variables y^μ and $\dot{y}^\kappa$, equalities (1.2) are satisfied along the solutions $q^k = q^k(t)$ of the differential dynamic equations of the system

$$\begin{aligned} \ddot{q}^k &= p^k + f^{k\kappa}\lambda_\kappa^{(1)} + q^{k(2)} \\ p^k &= m^k + f^{k\kappa}\lambda_\kappa^{(0)}, \quad q^{\kappa(2)} = m^{k(2)} + f^{k\kappa}\lambda_\kappa^{(0)} \end{aligned} \tag{3.3}$$

provided that they were satisfied for the initial conditions (1.3).

4. Stability

A necessary condition for the constraints (1.2) to be stable is the asymptotic stability of the corresponding integral manifold of system (3.3). We will represent Eqs. (3.3) and (2.20) in the form of systems of first-order differential equations in the variables q^i, q'^j and y^μ, y'^κ

$$\begin{aligned} \dot{q}^k &= q'^k, \quad \dot{q}'^k = p^k + f^{k\kappa}\lambda_\kappa^{(1)} + q^{k(2)} \\ p^k &= p^k(q^i, q'^j, t), \quad f^{k\kappa} = f^{k\kappa}(q^i, q'^j, t) \\ \lambda_\kappa^{(1)} &= \lambda_\kappa^{(1)}(q^i, q'^j, y^\mu, y'^\kappa, t), \quad q^{k(2)} = q^{k(2)}(q^i, q'^j, y^\mu, y'^\kappa, t) \end{aligned} \tag{4.1}$$

$$\begin{aligned} \dot{y}^\mu &= y'^\mu, \quad \dot{y}'^\kappa = b_\eta^\kappa y'^\eta + k_\nu^\kappa y^\nu + Y^{\kappa(2)} \\ b_\eta^\kappa &= b_\eta^\kappa(q^i, q'^j, t), \quad k_\nu^\kappa = k_\nu^\kappa(q^i, q'^j, t), \quad Y^{\kappa(2)} = Y^{\kappa(2)}(q^i, q'^j, y^\mu, y'^\eta, t) \end{aligned} \tag{4.2}$$

$$y^\mu = f^\mu(q^i, t), \quad y'^\mu = \partial_s f^\mu q'^s + \partial_t f^\mu, \quad y'^\rho = f^\rho(q^i, q'^j, t) \tag{4.3}$$

By virtue of equalities (4.3), the stability of the integral manifold (1.2) in the new variables can be treated as stability with respect to a part of the variables[7] of the system of equations (4.1) and (4.2). If, in the space of the variables q^i and q'^j, the distance up to the integral manifold (1.2) is determined by the quantity

$$\|z\|, \quad z = (z^1, \ldots, z^{m+r}), \quad z^\mu = y^\mu, \quad z^{m+\kappa} = y'^\kappa$$

then the stability of the manifold (1.2) can be judged from the stability properties of the trivial solution $y^\mu = 0$, $y'^\kappa = 0$ of system (4.2). The method of Lyapunov functions[8] can be used to investigate the stability of the trivial solution. If the function $V = V(q^s, q'^j, y^\mu, y'^\kappa, t)$, $V(q^s, q'^j, 0, 0, t) = 0$ is positive definite with respect to the variables y^μ and y'^κ, that is $V(q^s, q'^j, y^\mu, y'^\kappa, t) > 0$ when $y^\mu \delta_{\mu\nu} y^\nu + y'^\kappa \delta_{\kappa\nu} y'^\nu > 0$, $t \geq t_0$, and its derivative

$$\dot{V} = \partial_i V q'^i + \partial'_k V p^k + \partial_\mu V y'^\mu + \partial'_k V f^{k\kappa}\lambda_\kappa^{(1)} + \partial'_\kappa V Y^{\kappa(2)} + \partial'_k V q^{k(2)} + \partial_t V$$

$$\partial_i V \triangleq \frac{\partial V}{\partial q^i}, \quad \partial_\mu V \triangleq \frac{\partial V}{\partial y^\mu}, \quad \partial'_k V \triangleq \frac{\partial V}{\partial q'^k}, \quad \partial'_\kappa V \triangleq \frac{\partial V}{\partial y'^\kappa}, \quad \partial_t V \triangleq \frac{\partial V}{\partial t}$$

calculated on the basis of Eqs. (4.1) and (4.2), is a negative definite function, and the functions V, y^μ and y'^κ allow of an infinitesimal upper limit,[8] then the integral manifold (1.2) of system (4.1) is asymptotically stable.

The positive definite quadratic form in the variables y^μ and y'^κ

$$2V = u_{\mu\nu} y^\mu y^\nu + 2 v_{\mu\kappa} y^\mu y'^\kappa + w_{\kappa\eta} y'^\kappa y'^\eta \tag{4.4}$$

can be used as the Lyapunov function.

The coefficients $u_{\mu\nu}$, $\upsilon_{\mu\kappa}$ and $w_{\kappa\eta}$ of the form (4.4) are assumed to be continuous, differentiable, bounded functions of the variables q^i, q'^j and t over the whole domain of their variation

$$c_1 \le u_{\mu\nu}(q^i, q'^j, t) \ge c_2, \quad c_1 \le \upsilon_{\mu\kappa}(q^i, q'^j, t) \le c_2, \quad c_1 \le w_{\kappa\eta}(q^i, q'^j, t) \le c_2$$

The derivative of the function (4.4) reduces to the form

$$\dot{V} = u'_{\mu\nu}y^\mu y^\nu + \upsilon'_{\mu\kappa}y^\mu y'^\kappa + w'_{\kappa\eta}y'^\kappa y'^\eta + \dot{V}^{(3)} \tag{4.5}$$

$$2u'_{\mu\nu} = \partial_k u_{\mu\nu}q'^k + \partial'_k u_{\mu\nu}m^k + \partial_t u_{\mu\nu} + 2\upsilon_{\mu\kappa}k^\kappa_\nu \tag{4.6}$$

$$\upsilon'_{\mu\kappa} = u_{\mu\kappa} + \partial_k\upsilon_{\mu\kappa}q^k + \partial'_k\upsilon_{\mu\kappa}m^k + \partial_t\upsilon_{\mu\kappa} + \upsilon_{\mu\eta}b^\eta_\kappa + k^\eta_\mu w_{\eta\kappa} \tag{4.7}$$

$$\begin{gathered} 2w'_{\kappa\eta} = 2\upsilon_{\kappa\eta} + \partial_k w_{\kappa\eta}q'^k + \partial'_k w_{\kappa\eta}m^k + \partial_t w_{\kappa\eta} + 2w_{\kappa\xi}b^\xi_\eta \\ u_{\mu\kappa} = 0, \quad \upsilon_{\kappa\eta} = 0, \quad k^\kappa_\nu = 0, \quad \kappa > m \end{gathered} \tag{4.8}$$

$$\begin{gathered} \dot{V}^{(3)} = V'_k f^{k\kappa} l_{\kappa\eta}(k^\eta_\mu y^\mu + b^\eta_\xi y'^\xi) + (\upsilon_{\mu\kappa}y^\mu + w_{\eta\kappa}y'^\eta)Y^{\kappa(2)} \\ 2V'_k = \partial'_k u_{\mu\nu}y^\mu y^\nu + 2\partial'_k\upsilon_{\mu\kappa}y^\mu y'^\kappa + \partial'_k w_{\kappa\eta}y'^\kappa y'^\eta \end{gathered} \tag{4.9}$$

In the case when

$$\partial'_k u_{\mu\nu} = 0, \quad \partial'_k\upsilon_{\mu\nu} = 0, \quad \partial'_k w_{\mu\nu} = 0, \quad T^{(3)} = 0, \quad P^{(3)} = 0, \quad D^{(3)} = 0$$

the equality $\dot{V}^{(3)} = 0$ is satisfied as a consequence of relations (2.21) and (4.9), and expressions (4.6)–(4.8) take the form

$$2u'_{\mu\nu} = \partial_k u_{\mu\nu}q'^k + \partial_t u_{\mu\nu} + 2\upsilon_{\mu\kappa}k^\kappa_\nu$$

$$\upsilon'_{\mu\kappa} = u_{\mu\kappa} + \partial_k\upsilon_{\mu\kappa}q'^k + \partial_t\upsilon_{\mu\kappa} + \upsilon_{\mu\eta}b^\eta_\kappa + k^\eta_\mu w_{\eta\kappa}$$

$$2w'_{\kappa\eta} = 2\upsilon_{\kappa\eta} + \partial_k w_{\kappa\eta}q'^k + \partial_t w_{\kappa\eta} + 2w_{\kappa\xi}b^\xi_\eta$$

If, in expressions (2.4)–(2.6) for T, P and D, the coefficients m_{ij}, $a_{\kappa\eta}$, $k_{\mu\nu}$ and $c_{\kappa\eta}$ are constant and $T^{(3)} = 0$, $P^{(3)} = 0$, $D^{(3)} = 0$, then $Y^{\kappa(2)} = 0$ and system (4.2) consists of linear differential equations with constant coefficients. In this case, the stability of its trivial solution can be judged from the roots of the characteristic equation $\det(\mu^2\delta_\kappa - \mu b^\eta_\kappa - k^\eta_\kappa) = 0$

5. Numerical solution

For all $t > t_0$, the solution of the dynamic equations of system (4.1) satisfies the equalities

$$f^\mu(q^i, t) = 0, \quad \partial_s f^\mu q'^s + \partial_t f^\mu = 0, \quad f^\rho(q^i, q'^j, t) = 0 \tag{5.1}$$

if these equalities are satisfied when $t = t_0$. Stabilization of the constraints (5.1) during the numerical solution of system (4.1) can be achieved by an appropriate choice of the coefficients b^κ_η and k^κ_μ on the right-hand sides of the equations for the perturbations of the constraints (4.2). The conditions which are sufficient for stabilizing the constraints (5.1) are determined constructing a difference equation which is used for its solution.

Suppose

$$\Delta q^i(s) = q^i(s+1) - q^i(s), \quad q^i(s) = q^i(t_0 + \tau s), \quad \tau = \text{const}$$

$$q^i(0) = q^i_0, \quad q^i(s+1) = q^i(s) + \tau q'^i(s)$$

$$\dot{q}^i(0) = q'^i_0, \quad q'^i(s+1) = q'^i(s) + \tau(p^i(s) + f^{i\kappa}(s)\lambda^{(1)}_\kappa(s))$$

If, when $t = t_0 + \tau s$, the function

$$V(s) = V(q^i(s), q'^j(s), y^\mu(s), y'^\kappa(s), s)$$

is used to estimate the deviation of the solutions of system (4.1) from the manifold specified by the constraint Eq. (5.1), the quantity $V(s+1)$ can be defined by the expansion in series

$$\begin{aligned} V(s+1) &= V(s) + \partial_i V(s)\Delta q^i(s) + \partial'_j(s)\Delta q'^j(s) + \\ &+ \partial_\mu V(s)\Delta y^\mu(s) + \partial'_\kappa V(s)\Delta y'^\kappa(s) + \tau\partial_t V(s) + V^{(2)}(s) \end{aligned} \tag{5.2}$$

where $V^{(2)}(s)$ is the set of terms no lower than the second order in the variables $\Delta q^i(s)$, $\Delta q'^j(s)$, $\Delta y^\mu(s)$, $\Delta y'^\kappa(s)$ and τ. The increments $\Delta y^\mu(s)$ and $\Delta y'^\kappa(s)$ are defined by the equalities

$$\Delta y^\mu(s) = \tau y'^\mu(s), \quad \Delta y'^\kappa(s) = \tau(b^\kappa_\eta(s)y'^\eta(s) + k^\kappa_\mu(s)y^\mu(s)) \tag{5.3}$$

Taking relations (4.1), (4.2) and (5.3) into account, expression (5.2) reduces to the form

$$\begin{aligned} V(s+1) &= V(s) + \tau\dot{V}(s) + \tilde{V}(s) \\ \tilde{V}(s) &= V^{(2)}(s) - \tau(\partial'_j V(s) q^{j(2)}(s) + \partial'_\kappa V(s) Y^{\kappa(2)}(2)) \end{aligned} \tag{5.4}$$

On estimating the right-hand side of equality (5.4), it is possible to formulate the following assertions.

Theorem 1. *If the initial values of q_i^0 and q'^j_0 satisfy the condition*

$$\begin{aligned} &\|z_0\| \le \varepsilon, \quad z_0 = (z_0^1, \ldots, z_0^{m+r}) \\ &z_0^\mu = f^\mu(q_0^i, t_0), \quad z_0^{m+\mu} = \partial_k f^\mu(q_0^i, t_0) q'^k_0 + \partial_t f^\mu(q_0^i, t_0), \quad z_0^{m+\rho} = f^\rho(q_0^i, q'^j_0, t_0) \end{aligned} \tag{5.5}$$

and the restrictions

$$\begin{aligned} &l_1\|z(s)\|^2 \le V(s), \quad V(s) + \tau\dot{V}(s) \le \alpha l_1\|z(s)\|^2, \quad \tilde{V}(s) \le (1-\alpha)l_1\varepsilon^2 \\ &z(0) = z_0, \quad l_1 > 0, \quad 0 < \alpha < 1 \end{aligned}$$

are satisfied for all $s = 0, 1, \ldots, S$, then the inequality

$$\|z(s)\| \le \varepsilon \tag{5.6}$$

will hold for all $s = 1, \ldots, S$.

The proof of the theorem follows immediately from relations (5.4)–(5.6). Actually, if, in satisfying the conditions of the theorem, inequality (5.6) holds for a certain value of s, then

$$\begin{aligned} &\|z(s+1)\|^2 \le \frac{1}{l_1}V(s+1) = \frac{1}{l_1}(V(s) + \tau\dot{V}(s) + \tilde{V}(s)) \le \\ &\le \alpha\|z(s)\|^2 + (1-\alpha)\varepsilon^2 \le \alpha\varepsilon^2 + (1-\alpha)\varepsilon^2 = \varepsilon^2 \end{aligned}$$

Theorem 2. *If the initial values satisfy condition (5.5) and the restrictions*

$$\begin{aligned} &l_1\|z(s)\|^2 \le V(s) \le l_2\|z(s)\|^2, \quad \dot{V}(s) \le -l'\|z(s)\|^2, \quad \tilde{V}(s) \le (1-\alpha)l_1\varepsilon^2 \\ &\tau l' < l_2 \le \alpha l_1 + \tau l', \quad l_2 \ge l_1 > 0, \quad 0 < \alpha < 1 \end{aligned} \tag{5.7}$$

are satisfied for all $s = 0, 1, \ldots, S$, then inequality (5.6) will be satisfied for any $s = 1, \ldots, S$.

In fact, if condition (5.6) is true for a certain value of s, it follows from relations (5.4) and (5.7) that

$$\|z(s+1)\|^2 \le \frac{1}{l_1}V(s+1) \le \frac{1}{l_1}(l_2\|z(s)\|^2 - \tau l'\|z(s)\|^2 + \tilde{V}(s)) \le$$

$$\le \frac{1}{l_1}((k_2 - \tau l')\varepsilon^2 + (1-\alpha)l_1\varepsilon^2) \le \alpha\varepsilon^2 + (1-\alpha)\varepsilon^2 = \varepsilon^2$$

Theorem 3. *If the function $V = V(q^i, q'^j, y^\mu, y'^\kappa, t)$ and its derivative $\dot{V}$, calculated using the system of equations (4.1), (4.2), satisfies the conditions*

$$\dot{V} = -pV, \quad p > 0, \quad l_1\|z(s)\|^2 \le V(s) \le l_2\|z(s)\|^2$$

and the restrictions

$$\|z_0\| \le \varepsilon\sqrt{\frac{l_1}{l_2}}, \quad 0 < 1 - \tau p < 1 - \alpha \le 1, \quad \tilde{V}(s) \le \alpha l_1$$

are satisfied, then inequality (5.6) will be satisfied for any $s = 1, 2, \ldots, S$.

Actually, in this case,

$$V(s+1) = (1 - \tau p)V(s) + \tilde{V}(s)$$

and the inequalities

$$V(s+1) \le (1-\alpha)V(s) + \alpha l_1\|z_0\|^2 \le (1-\alpha)V(s) + \alpha V(0)$$

mean that $V(s) \le V(0)$. Consequently,

$$\|z(s)\|^2 \le \frac{1}{l_1}V(s) \le \frac{1}{l_1}V(0) \le \frac{l_2}{l_1}\|z_0\| \le \varepsilon^2$$

If the values of the Lyapunov function and its derivative of the form of (4.4), (4.5)

$$2V(s) = g_{\chi\varsigma}z^\chi(s)z^\varsigma(s), \quad \dot{V}(s) = g'_{\chi\varsigma}z^\chi(s)z^\varsigma(s) + \dot{V}^{(3)}; \quad \chi, \varsigma = 1, \ldots, m+r$$

where

$$g_{\mu\nu} = u_{\mu\nu}, \quad g_{\mu\kappa} = \upsilon_{\mu\kappa}, \quad g_{\eta\kappa} = w_{\eta\kappa}; \quad g'_{\mu\nu} = u'_{\mu\nu}, \quad g'_{\mu\kappa} = \upsilon'_{\mu\kappa}, \quad g'_{\eta\kappa} = w'_{\eta\kappa}$$

are used as $V(s)$, then the conditions for stabilizing the constraints are given by the following theorem.

Theorem 4. *If the initial values satisfy condition (5.5), $\tilde{V}(s) \le (1-\alpha)l_1\varepsilon^2$ and the restrictions*

$$g_{\chi\zeta}(s)z^\chi(z)z^\gamma(s) \ge 2l_1\|z(s)\|, \quad \tilde{g}_{\chi\zeta}(s)z^\chi(s)z^\zeta(s) \ge \alpha l_1\|z(s)\|$$

$$2\tilde{g}_{\chi\zeta} = g_{\chi\zeta} + 2\tau g'_{\chi\zeta}$$

are satisfied for all $s = 0, 1, \ldots, S$, then inequality (5.6) will be satisfied for any $s = 1, 2, \ldots, S$.

The proof of Theorem 4 *also follows directly from the chain of inequalities*

$$\|z(s+1)\|^2 \le \frac{1}{l_1}V(s+1) \le \frac{1}{l_1}(V(s) + \tau\dot{V}(s) + \tilde{V}(s)) \le$$

$$\le \frac{1}{2l_1}\|(g_{\beta\gamma}(s) + 2\tau g_{\beta\gamma}(s))\|\|z(s)\|^2 + \frac{1}{l_1}\tilde{V}(s) \le \alpha\varepsilon^2 + (1-\alpha)\varepsilon^2 = \varepsilon^2$$

In the case when $g_{\chi\zeta} = const$,

$$\dot{z}^\beta = p^\beta_\gamma z^\gamma, \quad p^\mu_\nu = 0, \quad p^\mu_{m+\nu} = \delta^\mu_\nu, \quad p^\mu_{m+\rho} = 0, \quad p^{m+\kappa}_\nu = k^\kappa_\nu, \quad p^{m+\kappa}_{m+\eta} = b^\kappa_\eta$$

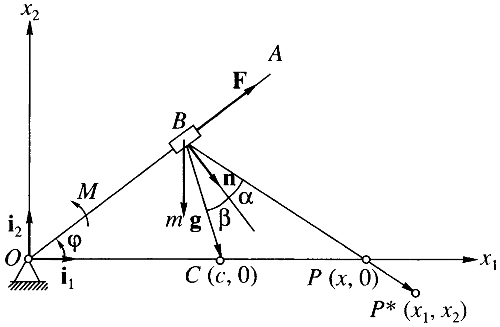

Fig. 1.

the expression $\tilde{g}_{\chi\zeta}$ *takes the form*

$$2\tilde{g}_{\chi\zeta} = g_{\chi\theta}(\delta_\zeta^\theta + 2\tau p_\zeta^\theta), \quad \theta = 1, 2, \ldots, m + r$$

and the following assertion is found to hold.

Theorem 5. *If the initial values* q_0^i *and* q'^j_0 *satisfy condition (5.5),* $z(0) = z_0$, $\tilde{V}(s) \le (1-\alpha)l_1\varepsilon^2$, *and the restrictions*

$$2l_1\|z(s)\|^2 \le g_{\beta\gamma}(s)z^\beta(s)z^\gamma(s), \quad \tilde{g}_{\beta\gamma}z^\beta(s)z^\gamma(s) \le \alpha l_1\|z(s)\|^2$$

$$2\tilde{g}_{\beta\gamma} = g_{\beta\xi}(\delta_\gamma^\xi + 2\tau p_\gamma^\xi)$$

are satisfied for all $s = 0, 1, \ldots, S$, *then inequality (5.6) will be satisfied for any* $s = 1, 2, \ldots, S$.

In fact, suppose condition (5.6) holds for a certain value of s. Then,

$$\|z(s+1)\|^2 \le \frac{1}{l_1}V(s+1) = \frac{1}{l_1}(V(s) + \tau\dot{V}(s) + \tilde{V}(s)) \le$$

$$\le \frac{1}{2l_1}g_{\beta\xi}(\delta_\gamma^\xi + 2\tau p_\gamma^\xi(s))z^\beta z^\gamma + \frac{1}{l_1}\tilde{V}(s) \le \alpha\varepsilon^2 + (1-\alpha)\varepsilon^2 = \varepsilon^2$$

6. Example. Control of an element of an adaptive optical system

An element of a discrete adaptive optical system[9] can be constructed with a mechanism consisting of a weightless crank OA which rotates about the Ox_3 axis and a slider B which is attached to it (Fig. 1). The position of the slider B is defined by the polar coordinates $q^1 = r$, $q^2 = \varphi$. The point $P^*(x_1, x_2)$: $x_1 = x_1(t)$, $x_2 = x_2(t)$, from which a ray of light emerges and is directed onto a mirror attached to the surface of the slider B, moves in the Qx_1x_2 plane. It is required to determine the magnitude F of the reaction $\mathbf{F}$ of the constraint

$$r - R = 0, \quad R = \text{const} \tag{6.1}$$

and the expression for the moment M applied to the crank for which the ray reflected from the mirror is incident at a fixed point $C(c, 0)$ of the Ox_1x_2 plane. The aim of the control is determined by the constraint equation

$$x(t)(2\cos\varphi - R/c) - R = 0 \tag{6.2}$$

where $x(t)$ is the value of the coordinate of the point P of the intersection of the line BP^* with the Ox_1 axis.

The slider B is treated as a point mass on which a gravitational force mg, in the opposite direction to the Ox_2 axis, acts. The excess variables y^1, y^2, $\dot{y}^1$ and $\dot{y}^2$ are defined by the relations

$$y^1 = r - R, \quad y^2 = x(t)(2\cos\varphi - R/c) - R \tag{6.3}$$

$$\dot{y}^1 = \dot{r}, \quad \dot{y}^2 = \dot{x}(t)(2\cos\varphi - R/c) - 2x(t)\dot{\varphi}\sin\varphi \tag{6.4}$$

In the case of the system considered

$$2T^0 = m(\dot{r}^2 + r^2\dot{\varphi}^2), \quad 2P^0 = mgr\sin\varphi, \quad D^0 = 0$$

Putting

$$2T = 2T^0 + (\dot{y}^1)^2 + (\dot{y}^2)^2, \quad 2P = 2P^0 + k(y^2)^2, \quad 2D = b(\dot{y}^2)^2; \quad k, b = \text{const}$$

the dynamic equations, corresponding to the mathematical model, can be represented by the system of differential equations

$$m\ddot{r} = mr\dot{\varphi}^2 - mg\sin\varphi + F, \quad mr^2\ddot{\varphi} = -m(2r\dot{r}\dot{\varphi} + gr\cos\varphi) + M \tag{6.5}$$

$$\ddot{y}^1 = 0, \quad \ddot{y}^2 = -b\dot{y}^2 - ky^2 \tag{6.6}$$

The expressions for the magnitude of the reactive force and the control moment on the right-hand sides of Eq. (6.5) are given by the equalities

$$\begin{gathered} F = \lambda_1\frac{\partial f^1}{\partial r} + \lambda_2\frac{\partial f^1}{\partial r} = \lambda_1, \quad M = \lambda_1\frac{\partial f^1}{\partial\varphi} + \lambda_2\frac{\partial f^2}{\partial\varphi} = -2\lambda_2 x(t)\sin\varphi \\ f^1 \equiv r - R, \quad f^2 \equiv x(t)(2\cos\varphi - R/c) - R \end{gathered} \tag{6.7}$$

If

$$r(0) = R = \text{const}, \quad \dot{r}(0) = 0$$

it follows from equality (6.3) and the first equation of (6.6) that $y^1(t) \equiv 0$ and $r(t) = R$. It then follows from Eq. (6.5) and the first equality of (6.7) that

$$\lambda_1 = m(g\sin\varphi - R\dot{\varphi}^2)$$

and the second equation of (6.6) takes the form

$$\ddot{\varphi} = -\frac{g}{R}\cos\varphi - \frac{2\lambda_2 x(t)}{mR^2}\sin\varphi \tag{6.8}$$

It remains to determine the multiplier λ_2. To do this, it suffices to differentiate the second equation of (6.4), taking account of the second equation of (6.6), Eqs. (6.8) and expressions (6.3) and (6.4). We have

$$\lambda_2 = \frac{mRL(\dot{\varphi}, \varphi, t)}{4x^2(t)\sin^2\varphi}$$

$$\begin{gathered} L(\dot{\varphi}, \varphi, t) = l_2(\varphi, t)\dot{\varphi}^2 + l_1(\varphi, t)\dot{\varphi} + l_0(\varphi, t) \\ l_2(\varphi, t) = 2Rx(t)\cos\varphi, \quad l_1(\varphi, t) = 2R(2\dot{x}(t) + bx(t))\sin\varphi \\ l_0(\varphi, t) = R(R/c - 2\cos\varphi)(\ddot{x}(t) + b\dot{x}(t) + kx(t)) - gx(t)\sin 2\varphi + kR^2 \end{gathered} \tag{6.9}$$

Substituting expression (6.9) into Eq. (6.8) we obtain an equation in φ

$$\ddot{\varphi} = -\frac{L(\dot{\varphi}, \varphi, t)}{2Rx(t)\sin\varphi} - \frac{g}{R}\cos\varphi \tag{6.10}$$

If the initial conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}_0(0) = \dot{\varphi}_0$ are chosen such that

$$\varphi_0 = \arccos\left(\frac{R}{2}\left(\frac{1}{c} + \frac{1}{x(0)}\right)\right), \quad \dot{\varphi}_0 = \frac{R\dot{x}(0)}{2x^2(0)\sin\varphi_0}$$

then the function

$$y = x(t)(2\cos\varphi - R/c) - R, \quad y \equiv y^2 \tag{6.11}$$

keeps the constant value $y=0$ in the corresponding solution of Eq. (6.10). It follows from the second equation of (6.6) that its trivial solution $y=0$ is asymptotically stable if $b>0$ and $k>0$. It remains to determine the magnitude of the coefficients b and k in order to ensure stabilization of the constraint (6.2) during the numerical solution of Eq. (6.10).

A numerical experiment was carried out using the following data

$$R = 3.4641, \quad c = 3, \quad m = 1, \quad \varphi_0 = 0.4685, \quad \dot{\varphi}_0 = -0.005, \quad x(t) = 2c + 0.5\cos t - 1$$

$$\varphi(s+1) = \varphi(s) + \tau\dot{\varphi}(s), \quad \tau = \text{const}$$

$$\dot{\varphi}(s+1) = \dot{\varphi}(s) - \frac{\tau}{R}\left(\frac{L(\dot{\varphi}(s), \varphi(s), s)}{2x(s)\sin\varphi(s)} + g\cos\varphi(s)\right)$$

Suppose that, when $t=\tau s$, the deviations of the solutions of Eq. (6.10) from the manifold prescribed by the constraint Eq. (6.2) are estimated by the quantity

$$\|z(s)\| = \sqrt{y^2(s) + y'^2(s)}, \quad z = (y, y'), \quad y' = \dot{y}, \quad \|z_0\| = 0.0412$$

If the function

$$V(s) = 3y^2(s) + 4y(s)y'(s) + 3y'^2(s), \quad y(s) = x(s)(2\cos\varphi(s) - R/c) - R$$

is used as an estimate for $||z(s)||$, then $l_1=1$ and $l_2=5$, and the equality $\dot{V}(s) = -pV(s)$ is satisfied when $p=b=-4/3$, $k=-1$.

We now expand the functions $V(s+1)$ in series

$$V(s+1) = V(s) + V_y(s)\Delta y(s) + V_{y'}(s)\Delta y'(s) + V^{(2)}(s)$$

Then,

$$V(s+1) = (1-\tau p)V(s) + \tilde{V}(s) \tag{6.12}$$

It follows from expression (6.15) that the inequality $V(s+1) \le V(s)$ is satisfied if $0 \le \alpha \le p\tau \le 1$, $\tilde{V}(s) \le \alpha l_1||z_0||^2$.

By putting $\tilde{V}(s) = (\tau^2/2)W(s)$ and taking only second-order terms in τ into account in expression (6.15), the estimate $W(s) \le 225$ can be obtained. Calculations carried out in accordance with Theorem 3 yield the following conditions for choosing the magnitude of τ

$$\tau_1 \le \tau \le \tau_2, \quad \tau_1 = 0.75\alpha, \quad \tau_2 = 0.0027\sqrt{2\alpha}$$

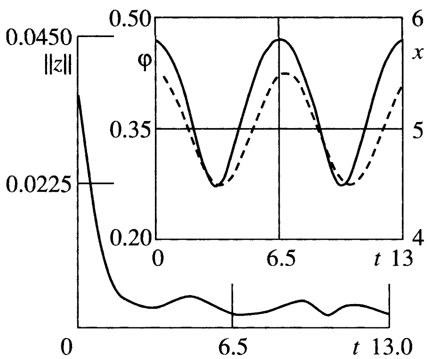

Fig. 2.

The condition $\tau_1 < \tau < \tau_2$ is satisfied if $0 < \alpha \leq 2.682 \times 10^{-5}$, and $\tau_1 = 0.1999 \times 10^{-5}$ and $\tau_2 = 0.6233 \times 10^{-5}$ correspond to the value $\alpha = 0.2665 \times 10^{-5}$. The inequality $||z_0|| \leq \varepsilon\sqrt{l_1/l_2}$ enables us to determine the restriction on the quantity ε: $\varepsilon \geq 9.213 \times 10^{-2}$.

Graphs of the functions $x = x(t)$, $\varphi = \varphi(t)$ and $||z(t)|| = \sqrt{y^2(t) + \dot{y}^2(t)}$ for $\tau = 2 \times 10^{-3}$ are shown in Fig. 2.

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